EXAMPLES OF MORI DREAM SPACES WITH PICARD NUMBER TWO

ATSUSHI ITO

Abstract. In this note, we give a sufficient condition such that a projective variety with Picard number two is a Mori dream space. Using this condition, we obtain examples of Mori dream spaces with Picard number two.

1. Introduction

Mori dream spaces, which were introduced by Hu and Keel in [HK], are special varieties which have very nice properties in view of the minimal model program. In the paper, Hu and Keel investigated properties of Mori dream spaces, especially those related to GIT.

We recall the definition of Mori dream spaces.

For a normal $\mathbb{Q}$-factorial projective variety X, we set

$$N^1(X) := (\mathrm{Cl}(X)/\equiv) \otimes \mathbb{Q} = (\mathrm{Pic}(X)/\equiv) \otimes \mathbb{Q}.$$

We denote by $\mathrm{Eff}(X)$, $\mathrm{Mov}(X)$, and $\mathrm{Nef}(X)$ the cones in $N^1(X)$ generated by effective, movable, and nef divisors respectively.

Definition 1.1. By a *small $\mathbb{Q}$-factorial modication (SQM)* of a projective variety X, we mean a birational map $f : X \dashrightarrow X'$ with X' projective and $\mathbb{Q}$-factorial, such that f is isomorphic in codimension one.

Definition 1.2. A normal projective variety X is called a *Mori Dream Space* if the following hold:

i) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$,
ii) $\mathrm{Nef}(X)$ is the affine hull of finitely many semiample line bundles,
iii) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies ii) and $\mathrm{Mov}(X) = \bigcup_i f_i^*(\mathrm{Nef}(X_i))$.

Quasi-smooth projective toric varieties are typical examples of Mori dream spaces. In [BCHM, Corollary 1.3.2], it is shown that $\mathbb{Q}$-factorial log Fano varieties are Mori dream spaces. See [AHL], [Jo], [TVV], etc. for other examples.

By definition, a normal $\mathbb{Q}$-factorial projective variety X is a Mori dream space if $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathbb{Q}$. Then, how about the case when the Picard number is two? The following theorem gives a sufficient condition for X to be a Mori dream space.

Theorem 1.3. *Let X be a normal $\mathbb{Q}$-factorial projective variety with Picard number 2 and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Assume that there exist nonzero effective Weil divisors $D_1, \ldots, D_r$ and $D'_1, \ldots, D'_{r'}$ on X for some $2 \leq r, r' \leq \dim X$ such that*

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a) $\mathrm{Cone}(D_1, \dots, D_r) \cap \mathrm{Cone}(D'_1, \dots, D'_{r'}) = \{0\}$,
b) $D_1 \cap \dots \cap D_r = D'_1 \cap \dots \cap D'_{r'} = \emptyset$.

Then X is a Mori dream space.

We will prove Theorem 1.3 by constructing SQMs using divisors D_i, D'_j. We explain the outline by an easy example.

Let $X \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(2, b)|$ be a general hypersurface on $\mathbb{P}^1 \times \mathbb{P}^3$ for $b \geq 1$. By using the defining equation of X, we can find effective divisors $D_1, D_2 \in |\mathcal{O}_X(-1, b)|$ and $D_3 \in |\mathcal{O}_X(0, 1)|$ on X such that $D_1 \cap D_2 \cap D_3 = \emptyset$. It is easy to check $D_1 \cap D_2$ is 1-dimensional and contracted by the morphism defined by D_3 (note $D_3 \sim \mathcal{O}_X(0, 1)$ is base point free). We consider the blowing up $\phi : \widetilde{X} \to X$ along $D_1 \cap D_2$ and the exceptional devisor $E \subset \widetilde{X}$. We can contract E to another direction and obtain a D_3-flop X^+. Then X^+ is a SQM of X and $D_1^+ \cap D_2^+ = \emptyset$, where D_1^+ and D_2^+ are the strict transforms of D_1 and D_2 on X^+ respectively. Since $D_1^+ \sim D_2^+$, $|D_1^+|$ is base point free and defines a morphism to a curve. Hence D_1^+ is neither ample nor big, which means $\mathbb{Q}_{\geq 0} D_1^+$ is an edge of $\mathrm{Nef}(X^+)$, $\mathrm{Mov}(X^+)$, and $\mathrm{Eff}(X^+)$. Thus $\mathbb{Q}_{\geq 0} D_1$ is an edge of $\mathrm{Mov}(X)$. Since $\mathcal{O}_X(1, 0)$ is base point free and not ample, $\mathbb{Q}_{\geq 0}\mathcal{O}_X(1, 0)$ is the other edge of $\mathrm{Mov}(X)$. Hence we have

$$\mathrm{Eff}(X) = \mathrm{Mov}(X) = \mathbb{Q}_{\geq 0}\mathcal{O}_X(1, 0) + \mathbb{Q}_{\geq 0}\mathcal{O}_X(-1, b) = \mathrm{Nef}(X) \cup \mathrm{Nef}(X^+)$$

as in Figure 1. We note that the movable cone $\mathrm{Mov}(X)$ of X is strictly larger than that of the ambient space $\mathbb{P}^1 \times \mathbb{P}^3$.

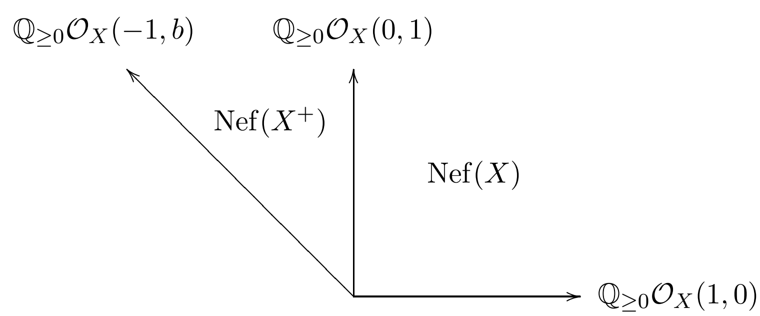

FIGURE 1.

As corollaries of Theorem 1.3, we obtain the following examples, which are special cases of Corollary 3.2. In both cases, we will construct divisors D_i, D'_j as in Theorem 1.3 by using the defining equations of X or Z. We note that Corollary 1.4 is proved by Oguiso in his private note [Og]. He also gives explicit descriptions of cones in $N^1(X)$ as Figure 1 for X in Corollary 1.4.

Corollary 1.4 ([Og])**.** *Let $X \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(a, b)|$ be a general hypersurface on $\mathbb{P}^1 \times \mathbb{P}^n$ such that $a, b > 0$ and $n \geq \max\{a, 3\}$. Then X is a Mori dream space.*

Corollary 1.5. *Let $Z \subset \mathbb{P}^N$ be a complete intersection of general hypersurfaces of degrees $d_1, \dots, d_s$, and $X \to Z$ be the blowing up at a general point $p \in Z$. If Z is a Fano variety, i.e., the anticanonical bundle $-K_Z = \mathcal{O}_Z(N + 1 - \sum_{i=1}^{s} d_i)$ is ample, then X is a Mori dream space.*

Remark 1.6. By Corollary 1.4, we can easily show that for any $n \geq 3$ and $\kappa \in \{-\infty, 0, 1, \ldots, n\}$, there exists an n-dimensional smooth Mori dream space with Picard number 2 whose Kodaira dimension is κ. See Remark 3.1 for detail.

Notations. Throughout this note, we work over the complex number field $\mathbb{C}$. A divisor means a Weil divisor. For a base point free divisor D, we denote by $\varphi_{|D|}$ the morphism defined by the complete linear system $|D|$. We define $\dim \emptyset = -1$.

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2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We will construct all the SQMs inductively by using the divisors D_i, D'_j.

Definition 2.1. Let X be a normal $\mathbb{Q}$-factorial projective variety with Picard number 2 and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Fix a (not necessarily extremal) ray $R \subset \mathrm{Eff}(X)$. For nonzero effective divisors D_1, D_2 on X, we denote

$$D_1 \succeq_R D_2 \quad \text{if} \quad D_2 \in \mathrm{Cone}(D_1, R).$$

When $R = \mathbb{Q}_{\geq 0} H$ for an effective divisor H, we write $D_1 \succeq_H D_2$ for short.

Roughly speaking, $D_1 \succeq_R D_2$ means that D_1 and D_2 are contained in the same side with respect to R and $\mathbb{Q}_{\geq 0} D_2$ is closer to R than $\mathbb{Q}_{\geq 0} D_1$.

Lemma 2.2. *Let X be a normal $\mathbb{Q}$-factorial projective variety with Picard number 2 and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Let A be an ample line bundle and $D_1 \succeq_A \cdots \succeq_A D_r$ be nonzero effective divisors on X such that $D_1 \cap \cdots \cap D_r = \emptyset$ and $2 \leq r \leq \dim X$. Then there exists $0 \leq k \leq r-1$ such that*

1) *D_{k+1} is semiample,*
2) *$D_{k+1} \notin \mathrm{Cone}(D_1, \cdots, D_k)$, $\mathrm{Cone}(D_1, \cdots, D_{k+1}) \cap \mathrm{Nef}(X) = \mathbb{Q}_{\geq 0} D_{k+1}$,*
3) *$\dim \varphi_{|mD_{k+1}|}(D_1 \cap \cdots \cap D_k) + k \leq \dim X - 1$ holds for sufficiently large and divisible m,*
4) *$D_1 \cap \cdots \cap D_k \neq \emptyset$.*

Remark 2.3. In the above lemma, we consider $\mathrm{Cone}(D_1, \ldots, D_k) = \{0\}$ and $D_1 \cap \cdots \cap D_k = X$ when $k = 0$.

Proof. Since $D_i \succeq_A D_r$ for $1 \leq i \leq r$, there exist rational numbers $a_i, b_i \geq 0$ such that $D_r \sim_{\mathbb{Q}} a_i D_i + b_i A$. For sufficiently large and divisible m, $\mathrm{Bs}(|mD_r|) \subset D_i$ since $mD_r \sim m a_i D_i + m b_i A$ and $m b_i A$ is base point free. Hence we have $\mathrm{Bs}(|mD_r|) \subset D_1 \cap \cdots \cap D_r = \emptyset$, which means D_r is semiample.

Let $0 \leq k_1 \leq r-1$ be the smallest i such that $\mathbb{Q}_{\geq 0} D_{i+1} = \mathbb{Q}_{\geq 0} D_r$. If $D_1 \cap \cdots \cap D_{k_1} \neq \emptyset$, it holds

$$\dim D_1 \cap \cdots \cap D_{k_1} \geq \dim X - k_1 \geq 1$$

by Krull's principal ideal theorem since D_i are $\mathbb{Q}$-Cartier. On the other hand, we have

$$\dim \varphi_{|mD_r|}(D_1 \cap \cdots \cap D_{k_1}) \leq r - k_1 - 1 \leq \dim X - k_1 - 1$$

for sufficiently large and divisible m since $D_{k_1+1}, \dots, D_r$ are linearly equivalent to positive rational multiples of D_r and $(D_1 \cap \cdots \cap D_{k_1}) \cap (D_{k_1+1} \cap \cdots \cap D_r) = \emptyset$. Hence $\varphi_{|mD_r|}$ contracts $D_1 \cap \cdots \cap D_{k_1}$. In particular, D_r is not ample. In this case, it is easy to check $k := k_1$ satisfies the statement of this lemma since $\mathbb{Q}_{\geq 0} D_{k+1} = \mathbb{Q}_{\geq 0} D_r$.

If $D_1 \cap \cdots \cap D_{k_1} = \emptyset$, we replace r with k_1 and repeat the above argument, and obtain $0 \leq k_2 \leq k_1 - 1$. Since $r > k_1 > k_2 > \cdots \geq 0$, this process must stop and we obtain k as in the statement of this lemma. □

Remark 2.4. In Lemma 2.2, $\varphi_{|mD_{k+1}|}$ is isomorphic on $X \setminus (D_1 \cap \cdots \cap D_k)$. In fact, if we write $D_{k+1} \sim_{\mathbb{Q}} a'_i D_i + b'_i A$, we have $a'_i, b'_i > 0$ for $1 \leq i \leq k$ by condition 2). Hence $\varphi_{|mD_{k+1}|}$ is isomorphic outside D_i for any $1 \leq i \leq k$ since $mb'_i A$ is very ample.

Since we will treat divisors satisfying conditions 1) $\sim$ 4) in Lemma 2.2 repeatedly, we define the following notation.

Definition 2.5. Let X be a normal $\mathbb{Q}$-factorial projective variety with Picard number 2 and $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. Let $D_1, \dots, D_{k+1}$ be nonzero effective divisors on X for $0 \leq k \leq \dim X - 1$. We say that $D_1, \dots, D_{k+1}$ satisfy condition $(*)$ if $D_1 \succeq_A \cdots \succeq_A D_{k+1}$ for an ample line bundle A and conditions 1) $\sim$ 4) in Lemma 2.2 are satisfied.

The following is the key proposition to construct SQMs inductively in the proof of Theorem 1.3.

Proposition 2.6. *Let X be a normal $\mathbb{Q}$-factorial projective variety with Picard number* 2 *and* $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)$. *Let $D_1, \dots, D_{k+1}$ be effective divisors on X which satisfy condition* $(*)$.

If $\dim D_1 \cap \cdots \cap D_k \leq \dim X - 2$, *there exists a SQM $X \dashrightarrow X^+$ and an integer* $0 \leq l \leq k-1$ *such that*

- D^+_{k+1} *is semiample,*
- $\mathrm{Nef}(X^+) = \mathbb{Q}_{\geq 0} D^+_{l+1} + \mathbb{Q}_{\geq 0} D^+_{k+1}$,
- $D^+_1, \dots, D^+_{l+1}$ *satisfy condition* $(*)$ *on* X^+,

where D^+_i is the strict transform of D_i on X^+ for each $1 \leq i \leq k+1$.

Proof. In Step 1, we construct a SQM X^+. In Step 2, we show the existence of l as in the statement.

Step 1. By 2) in condition $(*)$, we can write $D_k \sim_{\mathbb{Q}} a_i D_i + b_i D_{k+1}$ for some rational numbers $a_i > 0, b_i \geq 0$ for each $1 \leq i \leq k$. Fix a sufficiently large and divisible integer $m \in \mathbb{N}$ and take the scheme-theoretic intersection

$$C := \bigcap_{1 \leq i \leq k} m a_i D_i \subset X.$$

Let $\phi : \widetilde{X} \to X$ be the blowing up along C and E be the Cartier divisor on $\widetilde{X}$ such that $\phi^{-1} I_C = \mathcal{O}_{\widetilde{X}}(-E)$ for the ideal sheaf I_C of C.

Since $mD_k \sim ma_i D_i + mb_i D_{k+1}$ and $mb_i D_{k+1}$ is base point free, $\phi^*(mD_k) - E$ is base point free as well. Consider the following commutative diagram.

$$\begin{tikzcd}
& \widetilde{X} \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] \arrow[ddrr, bend left, "f"] & & \\
X \arrow[dr, "\pi"'] & & X' \arrow[dl, "\varpi'"] \arrow[dr, "\pi'"] & \\
& Z & & Z'.
\end{tikzcd}$$

In the above diagram, $\pi := \varphi_{|mD_{k+1}|}$, $f := \varphi_{|\phi^*(mD_k)-E|}$, X' is the image of $\widetilde{X} \xrightarrow{\pi\circ\phi\times f} Z \times Z'$, and ϖ' and π' are the first and second projections from $X' \subset Z \times Z'$ respectively.

By Remark 2.4, the restriction $\pi|_{X\setminus C} : X \setminus C \to Z \setminus \pi(C)$ is isomorphic. Since $\phi|_{\widetilde{X}\setminus E} : \widetilde{X} \setminus E \to X \setminus C$ is also isomorphic, the restriction of ϖ' on $X' \setminus \phi'(E)$ is an isomorphism onto $Z \setminus \pi(C)$. Hence $X \setminus C$ and $X' \setminus \phi'(E)$ are isomorphic.

By assumption, we have $\operatorname{codim}(C, X) \geq 2$ since $C = D_1 \cap \cdots \cap D_k$ as sets. To show that X and X' are isomorphic in codimension 1, we study $\phi'(E)$. By definition, there exists a natural surjection

$$\bigoplus_{i=1}^{k} \mathcal{O}_X(-ma_iD_i)|_C \to I_C/I_C^2.$$

Since $mD_k - ma_iD_i \sim mb_iD_{k+1}$, there exists an inclusion

$$E \hookrightarrow \mathbb{P}_C\left(\bigoplus_{i=1}^{k} \mathcal{O}_X(-ma_iD_i)|_C\right) \cong \mathbb{P}_C\Big(\bigoplus_{i=1}^{k} \mathcal{O}_X(mb_iD_{k+1})|_C\Big), \tag{2.1}$$

and we have

$$\mathcal{O}_{\widetilde{X}}(\phi^*mD_k - E)|_E \sim \mathcal{O}_{\mathbb{P}_C(\bigoplus_{i=1}^k \mathcal{O}_X(mb_iD_{k+1})|_C)}(1)|_E \tag{2.2}$$

under this inclusion.

From (2.1), (2.2), and the definition of ϕ', we have

$$\phi'(E) \hookrightarrow \mathbb{P}_{\pi(C)}\left(\bigoplus_{i=1}^{k} \mathcal{O}_Z(mb_i\overline{D}_{k+1})|_{\pi(C)}\right) \tag{2.3}$$

for a suitable scheme structure on $\pi(C)$, where $\overline{D}_{k+1}$ is the divisor on Z whose pullback on X is D_{k+1}. Hence

$$\begin{aligned}
\dim \phi'(E) &\leq \dim \mathbb{P}_{\pi(C)}\left(\bigoplus_{i=1}^{k} \mathcal{O}_Z(mb_i\overline{D}_{k+1})|_{\pi(C)}\right) \\
&= \dim \pi(C) + k - 1 \leq \dim X' - 2
\end{aligned}$$

by 3) in condition $(*)$ since $C = D_1 \cap \cdots \cap D_k$ as sets. Thus $\operatorname{codim}(\phi'(E), X') \geq 2$ holds. Since $X\setminus C \xrightarrow{\sim} Z\setminus\pi(C) \xleftarrow{\sim} X'\setminus\phi'(E)$, X and X' are isomorphic in codimension 1.

Let $\nu : X^+ \to X'$ be the normalization. Since $X' \setminus \phi'(E) \cong X \setminus C$ is normal, ν is isomorphic in codimension 1. Hence $X \dashrightarrow X^+$ is also isomorphic in codimension 1.

To show that X^+ is a SQM of X, it suffices to see that X^+ is $\mathbb{Q}$-factorial. Note that D_k^+ and D_{k+1}^+ are $\mathbb{Q}$-Cartier because they are the pullbacks of $\mathbb{Q}$-Cartier divisors

on Z' and Z respectively by the construction of X^+. Fix a prime divisor D^+ on X^+. Since D_k and D_{k+1} span $N^1(X) = \mathrm{Pic}(X)_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space, we can write $D \sim_{\mathbb{Q}} aD_k + bD_{k+1}$ for some $a, b \in \mathbb{Q}$, where D is the strict transform of D^+ on X. Since $X \dashrightarrow X^+$ is isomorphic in codimension 1, we have $D^+ \sim_{\mathbb{Q}} aD_k^+ + bD_{k+1}^+$, which means D^+ is $\mathbb{Q}$-Cartier.

Step 2. By the construction of X^+, D_k^+ and D_{k+1}^+ are semiample since they are the pullbacks of ample divisors on Z' and Z respectively. We show that there exists $0 \leq l \leq k-1$ as in the statement of this proposition.

First, we show $D_1^+ \cap \cdots \cap D_k^+ = \emptyset$. Since $\nu(D_i^+) = D_i'$, it is enough to show $D_1' \cap \cdots \cap D_k' = \emptyset$. By Step 1, $X \setminus (D_1 \cap \cdots \cap D_k) = X \setminus C \cong X' \setminus \phi'(E)$. Hence we have $(D_1' \cap \cdots \cap D_k') \setminus \phi'(E) = \emptyset$. Therefore it suffices to see $(D_1' \cap \cdots \cap D_k') \cap \phi'(E) = \emptyset$.

For $1 \leq i \leq k$, we have a divisor H_i on $\mathbb{P}_{\pi(C)}\left(\bigoplus_{j=1}^k \mathcal{O}_Z(mb_j\overline{D}_{k+1})|_{\pi(C)}\right)$ by the natural projection

$$\bigoplus_{j=1}^{k} \mathcal{O}_Z(mb_j\overline{D}_{k+1}) \to \bigoplus_{j \neq i} \mathcal{O}_Z(mb_j\overline{D}_{k+1}).$$

By construction, $D_i' \cap \phi'(E) = H_i \cap \phi'(E)$ holds under the inclusion (2.3). Since $\bigcap_{i=1}^k H_i = \emptyset$, we have

$$\bigcap_{i=1}^{k} D_i' \cap \phi'(E) = \bigcap_{i=1}^{k} H_i \cap \phi'(E) = \emptyset.$$

Hence $D_1' \cap \cdots \cap D_k'$ is empty, and so is $D_1^+ \cap \cdots \cap D_k^+$.

Since D_k^+ and D_{k+1}^+ are semiample, we can take an ample line bundle A^+ on X^+ such that $A^+ \in \mathrm{Cone}(D_k^+, D_{k+1}^+)$. Then we have $D_1^+ \succeq_{A^+} \cdots \succeq_{A^+} D_k^+$. Since $D_1^+ \cap \cdots \cap D_k^+ = \emptyset$, we can apply Lemma 2.2 to $D_1^+, \ldots, D_k^+$ and obtain $0 \leq l \leq k-1$ such that $D_1^+, \ldots, D_{l+1}^+$ satisfy condition $(*)$. In particular, D_{l+1}^+ is semiample but not ample. Since D_{k+1}^+ is semiample but not ample as well, we have $\mathrm{Nef}(X^+) = \mathbb{Q}_{\geq 0}D_{l+1}^+ + \mathbb{Q}_{\geq 0}D_{k+1}^+$. $\square$

Proof of Theorem 1.3. First, we show that if we renumber the indexes of $D_1, \ldots, D_r$ and $D_1', \ldots, D_{r'}'$ if necessary, there exist integers $0 \leq k \leq r-1$, $0 \leq k' \leq r'-1$ such that $D_1, \ldots, D_{k+1}$ and $D_1', \ldots, D_{k'+1}'$ satisfy condition $(*)$ respectively. In particular, D_{k+1} and $D_{k'+1}'$ are semiample and $\mathrm{Nef}(X) = \mathbb{Q}_{\geq 0}D_{k+1} + \mathbb{Q}_{\geq 0}D_{k'+1}'$. Hence X satisfies ii) in Definition 1.2.

Let A be an ample line bundle on X. Assume $A \notin \mathrm{Cone}(D_1, \ldots, D_r)$. By renumbering the indexes if necessary, we can apply Lemma 2.2 to $D_1, \ldots, D_r$ by the assumption b). Hence there exists $0 \leq k \leq r-1$ such that $D_1, \ldots, D_{k+1}$ satisfy condition $(*)$. Since D_{k+1} is nef and not contained in $\mathrm{Cone}(D_1', \ldots, D_{r'}')$, there exists an ample line bundle $A' \notin \mathrm{Cone}(D_1', \ldots, D_{r'}')$ by the assumption a). Applying Lemma 2.2 to $D_1', \ldots, D_{r'}'$, we obtain $0 \leq k' \leq r'-1$ such that $D_1', \ldots, D_{k'+1}'$ satisfy condition $(*)$. Since $\mathbb{Q}_{\geq 0}D_{k+1} \neq \mathbb{Q}_{\geq 0}D_{k'+1}'$, we have $\mathrm{Nef}(X) = \mathbb{Q}_{\geq 0}D_{k+1} + \mathbb{Q}_{\geq 0}D_{k'+1}'$.

If $A \in \mathrm{Cone}(D_1, \ldots, D_r)$, then $A \notin \mathrm{Cone}(D_1', \ldots, D_{r'}')$ by the assumption a). Hence we can apply the same argument.

To prove that X is a Mori dream space, it is enough to show that X satisfies iii) in Definition 1.2. In the rest of the proof, we investigate the edges of $\mathrm{Mov}(X)$.

Case 1. First, we consider the case $\dim D_1 \cap \cdots \cap D_k = \dim X$, i.e., $k = 0$. In this case, $D_1 = D_{k+1}$ is semiample. For a sufficient large and divisible m, we have $\dim \varphi_{|mD_1|}(X) = \dim \varphi_{|mD_1|}(\bigcap_{i=1}^{k} D_i) \leq \dim X - 1$ by 3) in condition $(*)$. Hence D_1 is not big. Thus $\mathbb{Q}_{\geq 0} D_1$ is an edge of both $\mathrm{Eff}(X)$ and $\mathrm{Mov}(X)$.

Case 2. Next, we consider the case $\dim D_1 \cap \cdots \cap D_k = \dim X - 1$. By Remark 2.4, $\pi := \varphi_{|mD_{k+1}|}$ is a birational morphism for sufficiently large and divisible m. Furthermore $\dim \pi(D_1 \cap \cdots \cap D_k) \leq \dim X - 2$ by 3) in condition $(*)$ since $k \geq 1$. Thus any prime divisor E on X contained in $D_1 \cap \cdots \cap D_k$ is contracted by π. We show that $\mathbb{Q}_{\geq 0} D_{k+1}$ and $\mathbb{Q}_{\geq 0} E$ are edges of $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$ respectively.

Any prime divisor on X contracted by π is $\mathbb{Q}$-linearly equivalent to aE for some $a > 0$ since X is $\mathbb{Q}$-factorial, $N^1(X) = \mathrm{Pic}(X)_{\mathbb{Q}}$, and the Picard number is 2. On the other hand, E is not movable since E is an exceptional divisor of π. Hence E is the unique exceptional divisor of π.

Let Z be the image of X by π. Since π is a divisorial contraction, Z is $\mathbb{Q}$-factorial and $\mathrm{Pic}(Z)_{\mathbb{Q}}$ is spanned by $\pi_* D_{k+1}$. For a prime divisor $D \neq E$ on X, $\pi(D)$ is a prime divisor since D is not contracted by π. Thus we have $\pi^*\pi_*(D) \sim_{\mathbb{Q}} b\pi^*\pi_* D_{k+1} \sim bD_{k+1}$ for some $b > 0$. Since $\pi^*\pi_*(D) - D$ is effective and its support is contained in the exceptional locus of π, $\pi^*\pi_*(D) - D \sim_{\mathbb{Q}} aE$ for some $a \geq 0$. Thus $D \sim_{\mathbb{Q}} -aE + bD_{k+1}$. Hence $\mathbb{Q}_{\geq 0} E$ is an edge of $\mathrm{Eff}(X)$ and any movable divisor is contained in $-\mathbb{Q}_{\geq 0} E + \mathbb{Q}_{\geq 0} D_{k+1}$. Thus $\mathbb{Q}_{\geq 0} D_{k+1}$ is an edge of $\mathrm{Mov}(X)$.

Case 3. We consider the case $\dim D_1 \cap \cdots \cap D_k \leq \dim X - 2$. Then we can apply Proposition 2.6 to $D_1, \ldots, D_{k+1}$ and obtain a SQM X^+ and $D_1^+, \ldots, D_{l+1}^+$ as in Proposition 2.6 for some $0 \leq l \leq k-1$. Set $k_1 = l$. If $\dim D_1^+ \cap \cdots \cap D_{k_1}^+ \geq \dim X^+ - 1$, we are reduced to Cases 1 or 2. Hence $\mathbb{Q}_{\geq 0} D_{k_1+1}$ is an edge of $\mathrm{Mov}(X) = \mathrm{Mov}(X^+)$.

If $\dim D_1^+ \cap \cdots \cap D_{k_1}^+ \leq \dim X^+ - 2$, we can apply Proposition 2.6 to $D_1^+, \ldots, D_{k_1+1}^+$ and obtain another SQM X^{++} and $D_1^{++}, \ldots, D_{k_2+1}^{++}$ for some $0 \leq k_2 \leq k_1 - 1$. Since $k > k_1 > k_2 > \cdots \geq 0$ is a decreasing sequence of non-negative integers, we reach Cases 1 or 2 after repeating this process finite time and obtain an edge of $\mathrm{Mov}(X)$.

Applying the same argument to $D_1', \cdots, D_{k'+1}'$, we obtain another edge of $\mathrm{Mov}(X)$. Then $\mathrm{Mov}(X)$ is the union of nef cones of SQMs constructed as above. Since the nef cone of each SQM is spanned by two semiample divisors, X satisfies iii) in Definition 1.2. $\square$

3. Examples

Corollaries 1.4, 1.5 are special cases of Corollary 3.2. To show Corollary 3.2, we give divisors satisfying conditions a), b) in Theorem 1.3 explicitly. To clarify the idea, we prove Corollary 1.4 first.

Proof of Corollary 1.4. Since $\dim X = n \geq 3$, $\mathrm{Pic}(X) \cong \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^n)$ holds by the Lefschetz hyperplane theorem. We denote by $\mathcal{O}_X(k,l)$ the restriction of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^n}(k,l)$ on X.

Let u, v be homogeneous coordinates on $\mathbb{P}^1$. Since $H^0(\mathbb{P}^1\times\mathbb{P}^n, \mathcal{O}(a,b)) = H^0(\mathbb{P}^1, \mathcal{O}(a))\otimes H^0(\mathbb{P}^n, \mathcal{O}(b))$, X is the zero section of

$$u^a f_0 + u^{a-1}vf_1 + \cdots + v^a f_a$$

for some general $f_i \in H^0(\mathbb{P}^n, \mathcal{O}(b))$. Set $Z := (f_0 = \cdots = f_a = 0) \subset \mathbb{P}^n$. Since f_i are general, $\dim Z = n - a - 1 \geq -1$.

Let $\overline{D}_i \sim \mathcal{O}_X(i-1, b)$ be the effective divisor on X defined by

$$(u^{i-1}f_0 + u^{i-2}vf_1 + \cdots + v^{i-1}f_{i-1})|_X$$

for $1 \leq i \leq a$. Since

$$u^{a-i+1}(u^{i-1}f_0 + u^{i-2}vf_1 + \cdots + v^{i-1}f_{i-1})|_X = -v^i(u^{a-i}f_i + \cdots + v^{a-i}f_a)|_X,$$

$D_i := \overline{D}_i - (v^i = 0)|_X \sim \mathcal{O}_X(-1, b)$ is an effective divisor on X for each $1 \leq i \leq a$. By the definition of D_i, we have

$$D_1 \cap \cdots \cap D_a = (p_2^* f_0 = \cdots = p_2^* f_a = 0) = p_2^{-1}(Z) = \mathbb{P}^1 \times Z,$$

where $p_2 : \mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^n$ is the second projection. Choose general members $\overline{D}_{a+1}, \ldots, \overline{D}_n$ in $|\mathcal{O}_{\mathbb{P}^n}(1)|$ and set $D_i = (p_2^*\overline{D}_i)|_X$ for $a+1 \leq i \leq n$. Since $Z \cap \overline{D}_{a+1} \cap \cdots \cap \overline{D}_n = \emptyset$, we have $D_1 \cap \cdots \cap D_n = \emptyset$.

If D'_1, D'_2 are general members in $\mathcal{O}_X(1,0)$, we have $D'_1 \cap D'_2 = \emptyset$. Since $D_1, \ldots, D_n$ and D'_1, D'_2 satisfy a), b) in Theorem 1.3, X is a Mori dream space. □

Remark 3.1. Let $X \subset \mathbb{P}^k \times \mathbb{P}^{n+1-k}$ be a general hypersurface in $|\mathcal{O}(a,b)|$ for $n \geq 3, a, b > 0$. If $2 \leq k \leq n-1$, $\mathcal{O}_X(1,0)$ and $\mathcal{O}_X(0,1)$ are semiample and not big. Hence X is a Mori dream space since $\mathrm{Eff}(X) = \mathrm{Mov}(X) = \mathrm{Nef}(X) = \mathbb{Q}_{\geq 0}\mathcal{O}_X(1,0) + \mathbb{Q}_{\geq 0}\mathcal{O}_X(0,1)$. By choosing suitable k, a, b, we can easily check that the Kodaira dimension $\kappa(X)$ of X can be any number in $\{-\infty, 0, 1, 2, \cdots, n\}$ except 1.

On the other hand, a general hypersurface $X \subset \mathbb{P}^1 \times \mathbb{P}^n$ in $|\mathcal{O}(3, n+1)|$ for $n \geq 3$ is a Mori dream space with $\kappa(X) = 1$ by Corollary 1.4.

Hence for any $n \geq 3$ and $\kappa \in \{-\infty, 0, 1, \cdots, n\}$, there exists an n-dimensional smooth Mori dream space with $\kappa(X) = \kappa$ and Picard number 2. This is shown in [Og] when $n = 3$.

Corollary 1.4 can be generalized by a similar argument as follows.

Corollary 3.2. *Let Y be a smooth Mori dream space with Picard number* 1*, and A be a base point free line bundle on Y. Let $\pi : \mathbb{P} = \mathbb{P}_Y(\mathcal{O}_Y \oplus A) \to Y$ be the $\mathbb{P}^1$-bundle on Y and $\mathcal{O}_{\mathbb{P}}(1)$ be the tautological line bundle on $\mathbb{P}$. Let $X = D^1 \cap \cdots \cap D^s \subset \mathbb{P}$ be a complete intersection on $\mathbb{P}$ of general divisors $D^j \in |\mathcal{O}_{\mathbb{P}}(a_j) \otimes \pi^* B_j|$, where $a_1, \ldots, a_s$ are positive integers and $B_1, \ldots, B_s$ are ample and base point free line bundles on Y. If $\dim X \geq \max\{\sum_{j=1}^s a_j, 3\}$, X is a Mori dream space.*

Proof. Set $n = \dim X$. Since A is nef and B_j is ample, D^j is ample. Hence $\mathrm{Pic}(X) \cong \mathrm{Pic}\,\mathbb{P} = \mathbb{Z}\mathcal{O}_{\mathbb{P}}(1) \oplus \pi^* \mathrm{Pic}\, Y$ by the Lefschetz hyperplane theorem and $n \geq 3$.

Let $u \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ and $v \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* A^{-1})$ be the sections corresponding to the first and second summands of $\mathcal{O} \oplus A$ respectively.

Take $f^j \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a_j) \otimes \pi^* B_j)$ which defines D^j. Since

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a_j) \otimes \pi^* B_j) \cong H^0(Y, \mathrm{Sym}^{a_j}(\mathcal{O} \oplus A) \otimes B_j) \cong \bigoplus_{i=0}^{a_j} H^0(Y, A^{\otimes i} \otimes B_j),$$

we can write

$$f^j = u^{a_j} f_0^j + u^{a_j - 1} v f_1^j + \cdots + v^{a_j} f_{a_j}^j$$

for some $f_i^j \in H^0(Y, A^{\otimes i} \otimes B_j)$. Since X is general, each f_i^j is a general section of the base point free line bundle $A^{\otimes i} \otimes B_j$. For $1 \le i \le a_j$, we define a divisor $D_i^j \sim \mathcal{O}_{\mathbb{P}}(-1) \otimes \pi^*(A^{\otimes i} \otimes B_j)|_X$ on X to be

$$D_i^j = (u^{i-1} f_0^j + u^{i-2} v f_1^j + \cdots + v^{i-1} f_{i-1}^j = 0)|_X - (v^i = 0)|_X,$$

which is effective as in the proof of Corollary 1.4. Similar to the proof of Corollary 1.4, we have

$$\bigcap_{1 \le i \le a_j, 1 \le j \le s} D_i^j = \pi^{-1} \left(\bigcap_{0 \le i \le a_j, 1 \le j \le s} (f_i^j = 0) \right).$$

Since f_i^j are general and $\dim Y = n + s - 1$, it holds that

$$\dim \bigcap_{i,j} (f_i^j = 0) = \dim Y - \sum_j (a_j + 1) = n - \sum a_j - 1.$$

Choose general hypersurfaces $\overline{D}_1, \ldots, \overline{D}_{n - \sum a_j}$ on Y and set $D_i = (\pi^* \overline{D}_i)|_X$ for $1 \le i \le n - \sum a_j$. Since $\overline{D}_1 \cap \cdots \cap \overline{D}_{n - \sum a_j} \cap \bigcap_{i,j} (f_i^j = 0) = \emptyset$, we have

$$D_1 \cap \cdots \cap D_{n - \sum a_j} \cap \bigcap_{1 \le i \le a_j, 1 \le j \le s} D_i^j = \emptyset.$$

We set $D_1' = (u = 0)|_X \sim \mathcal{O}_{\mathbb{P}}(1)|_X$, $D_2' = (v = 0)|_X \sim \mathcal{O}_{\mathbb{P}}(1) \otimes \pi^* A^{-1}|_X$. Then we have $D_1' \cap D_2' = \emptyset$. Since $D_1, \ldots, D_{n - \sum a_j}, \{D_i^j\}_{i,j}$ and D_1', D_2' satisfy a), b) in Theorem 1.3, X is a Mori dream space. □

Proof of Corollary 1.5. When $\dim X = 2$, X is a Del Pezzo surface. Hence X is a Mori dream space.

Thus we may assume $\dim X = \dim Z \ge 3$. Let $Z \subset \mathbb{P}^{n+s}$ be a complete intersection of general hypersurfaces of degrees $d_1, \ldots, d_s$. For the blowing up $\mu : \widetilde{\mathbb{P}}^{n+s} \to \mathbb{P}^{n+s}$ at a general point $p \in Z$, X is a complete intersection on $\widetilde{\mathbb{P}}^{n+s}$ of general hypersurfaces in $|\mu^* \mathcal{O}(d_1) - E|, \ldots, |\mu^* \mathcal{O}(d_s) - E|$ since Z and p are general. Let

$$\pi : \widetilde{\mathbb{P}}^{n+s} = \mathbb{P}_{\mathbb{P}^{n+s-1}}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}^{n+s-1}$$

be the $\mathbb{P}^1$-bundle obtained from $|\mu^* \mathcal{O}(1) - E|$. Since $\mu^* \mathcal{O}(1)$ is the tautological bundle of π and $\mu^* \mathcal{O}(1) - E = \pi^* \mathcal{O}_{\mathbb{P}^{n+s-1}}(1)$, we can apply Corollary 3.2 to $Y = \mathbb{P}^{n+s-1}$, $A = \mathcal{O}_{\mathbb{P}^{n+s-1}}(1)$, $a_j = d_j - 1$, and $B_j = \mathcal{O}_{\mathbb{P}^{n+s-1}}(1)$ if $\dim X = n \ge \sum (d_j - 1)$. This condition is nothing but the ampleness of $-K_Z$ by the adjunction formula. □

Remark 3.3. If Z is not Fano, Corollary 1.5 does not hold in general. For example, X is not a Mori dream space if Z is a very general quartic surface in $\mathbb{P}^3$ and $p \in Z$ is a very general point by [AL, Proposition 6.3]. We note that Proposition 6.3 in [AL] claims that X is not a Mori dream space for *some* Z and p, but their proof works for very general Z, p.

By checking the proof of Theorem 1.3 carefully, we can explicitly describe cones in $N^1(X)$ for Corollaries 1.4, 1.5, or 3.2. We illustrate the description by a special case of Corollary 1.5. We leave the other cases for the reader.

Example 3.4. Let $Z \subset \mathbb{P}^{n+1}$ be a general hypersurface of degree $n+1$ for $n \geq 3$. Let $\mu : X \to Z$ be the blowing up at a general point $p \in Z$ and E be the exceptional divisor. We set $H = \mu^*\mathcal{O}_Z(1)$. By the proofs of Corollaries 1.5 and 3.2, we have effective divisors $D_i \sim iH - (i+1)E$ on X for $1 \leq i \leq n$ such that $D_1 \cap \cdots \cap D_n = \emptyset$. It is easy to see $D_1, \ldots, D_n$ satisfy condition $(*)$ (see the proof of Lemma 2.2), hence $\mathrm{Nef}(X) = \mathbb{Q}_{\geq 0}H + \mathbb{Q}_{\geq 0}D_n$ as in the first paragraph of the proof of Theorem 1.3. In this case, k in the proof of Theorem 1.3 is $n-1$.

Since $\dim D_1 \cap \cdots \cap D_{n-1} = 1 \leq \dim X - 2$, we obtain a SQM $X^{(2)}$ of $X^{(1)} := X$ by case 3 in the proof of Theorem 1.3. We denote by $D_i^{(j)}$ the strict transform of D_i on $X^{(j)}$ (Note that this $D_i^{(j)}$ is different from D_i^j in the proof of Corollary 3.2). By the proof of Proposition 2.3, k_1 in case 3 in the proof of Theorem 1.3 is $n-2$ in this case. Hence $\mathrm{Nef}(X^{(2)}) = \mathbb{Q}_{\geq 0}D_{n-1}^{(2)} + \mathbb{Q}_{\geq 0}D_n^{(2)}$ holds. By repeating this process, we obtain a SQM $X^{(j)}$ for each $2 \leq j \leq n-1$ such that $D_1^{(j)}, \ldots, D_{n+1-j}^{(j)}$ satisfy condition $(*)$ and $\mathrm{Nef}(X^{(j)}) = \mathbb{Q}_{\geq 0}D_{n+1-j}^{(j)} + \mathbb{Q}_{\geq 0}D_{n+2-j}^{(j)}$. On $X^{(n-1)}$, we reach case 2 in the proof of Theorem 1.3. Thus $D_1^{(n-1)}$ and $D_2^{(n-1)}$ are edges of $\mathrm{Eff}(X^{(n-1)}) = \mathrm{Eff}(X)$ and $\mathrm{Mov}(X^{(n-1)}) = \mathrm{Mov}(X)$ respectively.

From the above argument, we have the following description.

$$\begin{aligned}
\mathrm{Nef}(X) = \mathrm{Nef}(X^{(1)}) &= \mathbb{Q}_{\geq 0}H + \mathbb{Q}_{\geq 0}D_n, \\
\mathrm{Nef}(X^{(j)}) &= \mathbb{Q}_{\geq 0}D_{n+1-j} + \mathbb{Q}_{\geq 0}D_{n+2-j} \quad \text{for} \quad 2 \leq j \leq n-1, \\
\mathrm{Mov}(X) &= \bigcup_{j=1}^{n-1} \mathrm{Nef}(X^{(j)}) = \mathbb{Q}_{\geq 0}H + \mathbb{Q}_{\geq 0}D_2, \\
\mathrm{Eff}(X) &= \mathbb{Q}_{\geq 0}E + \mathbb{Q}_{\geq 0}D_1.
\end{aligned}$$

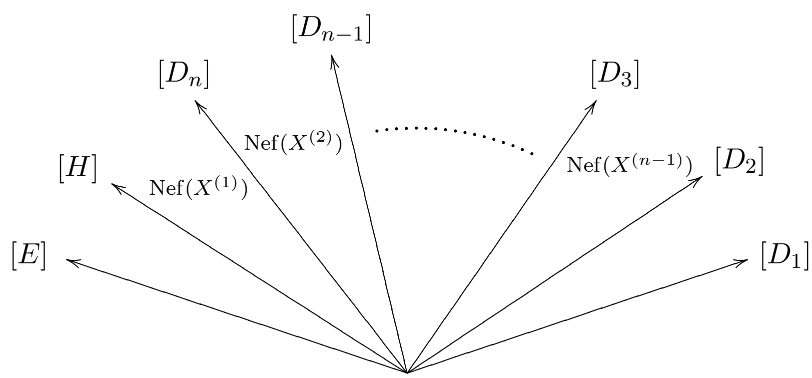

Figure 2.

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Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914, Japan.
E-mail address: itoatsu@ms.u-tokyo.ac.jp